

The Reconstruction Problem and Weak Quantum Values

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Abstract

Quantum Mechanical weak values are an interference effect measured by the cross-Wigner transform $\mathcal{W}(\phi, \psi)$ of the post- and pre-selected states, leading to a complex quasi-distribution $\rho_{\phi, \psi}(x, p)$ on phase space. We show that the knowledge of $\rho_{\phi, \psi}(z)$ and of one of the two functions ϕ, ψ unambiguously determines the other, thus generalizing a recent reconstruction result of Lundeen and his collaborators.

1 Introduction

In 1958 W. Pauli [22] mentions the problem of the reconstruction of a quantum state knowing its position and momentum; this conjecture was later disproved; see H. Reichenbach's book [23]; also Corbett [8] for a review of the Pauli problem. However, in [19] Lundeen and his coworkers consider the following experiment: a weak measurement of position is performed on a particle; thereafter a strong measurement of momentum is made. This allows them to effectively reconstruct the wavefunction $\psi(x)$ pointwise by scanning through all values of x . The aim of the present paper is to show that, more generally, the wavefunction can be reconstructed from the knowledge of a complex probability distribution $\rho_{\phi, \psi}(x, p)$ on phase space, related

to the notion of weak values, and expressed in the terms of the cross-Wigner transform $\mathcal{W}(\phi, \psi)$ of the pre- and postselected states. Our proof is based on the fact that the knowledge of $\mathcal{W}(\phi, \psi)$ and of one of the two functions ϕ, ψ allows to determine uniquely and unambiguously (in infinitely many ways) the other function. This is of course in strong contrast with the case of the usual Wigner distribution $\mathcal{W}\psi = \mathcal{W}(\psi, \psi)$ whose knowledge only determines ψ up to a factor with modulus one.

In what follows the position and momentum vectors are $x = (x_1, \dots, x_n)$ and $p = (p_1, \dots, p_n)$, respectively; $px = p_1x_1 + \dots + p_nx_n$ is their scalar product. We write $d^n x = dx_1 dx_2 \dots dx_n$, and all the integrations are performed over n -dimensional space \mathbb{R}^n . The \hbar -Fourier transform of a function ψ is defined by

$$F\psi(p) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}px} \psi(x) d^n x.$$

2 Weak Values and the Cross-Wigner Transform

Let \hat{A} be a quantum observable associated to a function A by the Weyl correspondence: $\hat{A} \xleftrightarrow{\text{Weyl}} A$ [9, 10, 18] and ϕ, ψ two non-orthogonal states. Aharonov and coworkers [1, 2, 3, 4, 5] have introduced and studied the notion of weak measurement and the weak value of an observable (also see [16]). The *weak value* of \hat{A} with respect to the pair (ϕ, ψ) is the complex number

$$\langle \hat{A} \rangle_{\text{weak}}^{\phi, \psi} = \frac{\langle \phi | \hat{A} | \psi \rangle}{\langle \phi | \psi \rangle}, \quad \langle \phi | \psi \rangle \neq 0. \quad (1)$$

According to

$$\langle \hat{A} \rangle_{\text{weak}}^{\phi, \psi} = \int A(x, p) \rho_{\phi, \psi}(x, p) d^n x d^n p \quad (2)$$

When $\phi = \psi$ this is the usual average value $\langle \hat{A} \rangle^\psi$ of \hat{A} in the state ψ . The physical interpretation of the weak value is the following (see Jozsa [17] and Parks and Gray [21] for concise and clear expositions): viewing $|\psi\rangle$ as a preselected state and $\langle\phi|$ as a postselected state, if we couple a measuring device whose pointer has position coordinate x to the system and subsequently measure that coordinate then the mean value of the pointer is

$$\langle \hat{x} \rangle = g \operatorname{Re} \langle \hat{A} \rangle_{\text{weak}}^{\phi, \psi}$$

if the coupling interaction is the standard von Neumann interaction Hamiltonian $\hat{H} = g\hat{A}p$. In addition the mean of the pointer momentum is given

by

$$\langle \hat{p} \rangle = 2 \frac{g}{\hbar} v \operatorname{Im} \langle \hat{A} \rangle_{\text{weak}}^{\phi, \psi}.$$

(This needs sufficiently weak interaction, see the discussions in Duck et al. [12] and in Parks et al. [20]).

In a recent work [11] we have shown that the weak value can be calculated by averaging A over the complex phase space function

$$\rho_{\phi, \psi}(x, p) = \frac{\mathcal{W}(\phi, \psi)(x, p)}{\langle \phi | \psi \rangle} \quad (3)$$

where

$$\mathcal{W}(\phi, \psi)(x, p) = \left(\frac{1}{2\pi\hbar} \right)^n \int e^{-\frac{i}{\hbar} p y} \phi^* \left(x + \frac{1}{2} y \right) \psi \left(x - \frac{1}{2} y \right) d^n y \quad (4)$$

is the cross-Wigner transform (we are conjugating ϕ and not ψ in order to be consistent with the bra-ket notation; in most mathematical texts the transform defined by (4) would be denoted by $\mathcal{W}(\psi, \phi)$). The function $\rho_{\phi, \psi}$ satisfies the marginal conditions

$$\int \rho_{\phi, \psi}(x, p) d^n p = \frac{\phi^*(x) \psi(x)}{\langle \phi | \psi \rangle}, \quad \int \rho_{\phi, \psi}(x, p) d^n x = \frac{[F\phi(p)]^* F\psi(p)}{\langle \phi | \psi \rangle}; \quad (5)$$

and hence in particular

$$\int \rho_{\phi, \psi}(x, p) d^n x d^n p = 1.$$

The function $\rho_{\phi, \psi}$ can thus be viewed as a complex quasi-probability density on phase space; its real and imaginary parts moreover satisfy

$$\int \operatorname{Re} \rho_{\phi, \psi}(x, p) d^n x d^n p = 1, \quad \int \operatorname{Im} \rho_{\phi, \psi}(x, p) d^n x d^n p = 0$$

The appearance of the cross-Wigner function is characteristic of interference phenomena, and suggests the following interpretation of weak values¹ [2, 3, 4]. Assume that we measure an observable \hat{A} at an initial time t_{in} and that a non-degenerate eigenvalue was found: $|\psi(t_{\text{in}})\rangle = |\hat{A} = a\rangle$ (the pre-selected state); similarly at a final time t_{fin} a measurement of another observable \hat{B} yields $|\phi(t_{\text{fin}})\rangle = |\hat{B} = b\rangle$ (the post-selected state). Choose now some

¹But this interpretation is far from being unanimously shared in the scientific community, and has led to an ongoing epistemological debate on the back-action of future on the past.

intermediate time $t : t_{\text{in}} < t < t_{\text{fin}}$. Following the time-symmetric approach to quantum mechanics, at this intermediate time the system is described by the *two* wavefunctions

$$\psi = \psi_t = U_{t,t_{\text{in}}}^{H_{\text{in}}} \psi(t_{\text{in}}) \quad , \quad \phi = \phi_t = U_{t,t_{\text{fin}}}^{H_{\text{fin}}} \phi(t_{\text{fin}}) \quad (6)$$

where $U_{t,t'}^H = e^{-i\hat{H}(t-t')/\hbar}$ is the Schrödinger unitary evolution operator if \hat{H} is the quantum Hamiltonian). Notice that ϕ_t travels *backwards* in time since $t < t_{\text{fin}}$. The situation is thus the following: at any time $t' < t$ the system under consideration is in the state $|\psi_{t'}\rangle = U_{t',t_{\text{in}}}^H |\psi(t_{\text{in}})\rangle$ and has Wigner distribution $\mathcal{W}\psi_{t'}$; at any time $t'' > t$ the system is in the state $|\phi_{t''}\rangle = U_{t'',t_{\text{fin}}}^H |\phi(t_{\text{fin}})\rangle$ and has Wigner distribution $\mathcal{W}\phi_{t''}$. But at time t it is the superposition $|\psi_t\rangle + |\phi_t\rangle$ of both states, and the Wigner distribution

$$\mathcal{W}(\phi_t + \psi_t) = \mathcal{W}(\phi_t + \psi_t, \phi_t + \psi_t)$$

of this state is

$$\mathcal{W}(\phi_t + \psi_t) = \mathcal{W}\phi_t + \mathcal{W}\psi_t + 2 \text{Re } \mathcal{W}(\phi_t, \psi_t). \quad (7)$$

This equality shows the emergence at time t of the interference term $2 \text{Re } \mathcal{W}(\phi_t, \psi_t) = 2 \text{Re } \mathcal{W}(\phi, \psi)$, signalling a strong interaction between the states $|\psi_t\rangle = |\psi\rangle$ and $|\phi_t\rangle = |\phi\rangle$.

3 The Reconstruction Problem

3.1 An example

As briefly explained in the Introduction Lundeen and his coworkers [19] consider the following experiment: a weak measurement of position is performed on a particle; this amounts to applying the projection operator $\hat{\Pi}_x = |x\rangle\langle x|$ to the pre-selected state $|\psi\rangle$, which yields $\hat{\Pi}_x|\psi\rangle = \psi(x)|x\rangle$. Thereafter a strong measurement of momentum is made, yielding a value p_0 ; the result of the weak measurement is thus

$$\langle \hat{\Pi}_x \rangle_{\text{weak}}^{\phi, \psi} = \frac{e^{\frac{i}{\hbar} p_0 x} \psi(x)}{F\psi(p_0)}. \quad (8)$$

This allows the reconstruction of the function ψ from $\langle \hat{\Pi}_x \rangle_{\text{weak}}^{\phi, \psi}$:

$$\psi(x) = k e^{-\frac{i}{\hbar} p_0 x} \langle \hat{\Pi}_x \rangle_{\text{weak}}^{\phi, \psi} \quad , \quad k = F\psi(p_0). \quad (9)$$

Let us retrieve this result using the Wigner formalism developed above. Obviously, the operator \hat{A} is here the projector $\hat{\Pi}_x$ whose analytical expression is given by

$$\hat{\Pi}_x \psi(y) = \psi(x) \delta(x - y). \quad (10)$$

Choosing for the post-selected state ϕ the normalized momentum wavefunction

$$\phi_{p_0}(x) = (2\pi\hbar)^{-n/2} e^{\frac{i}{\hbar} p_0 x}$$

we have

$$\begin{aligned} \mathcal{W}(\psi, \phi_{p_0})(x, p) &= \left(\frac{1}{2\pi\hbar}\right)^{3n/2} \int e^{-\frac{i}{\hbar} p y} \psi^*(x + \frac{1}{2}y) e^{\frac{i}{\hbar} p_0(x - \frac{1}{2}y)} d^n y \\ &= \left(\frac{1}{2\pi\hbar}\right)^{3n/2} e^{\frac{i}{\hbar} p_0 x} \int e^{-\frac{i}{\hbar} (p + \frac{1}{2}p_0)y} \psi^*(x + \frac{1}{2}y) d^n y \end{aligned}$$

which –after the change of variables $x' = x + \frac{1}{2}y$ and integrating– becomes

$$\mathcal{W}(\psi, \phi_{p_0})(x, p) = \left(\frac{1}{\pi\hbar}\right)^n e^{\frac{2i}{\hbar}(p - p_0)x} F\psi(2p - p_0).$$

Taking into account the fact that $\langle \phi_{p_0} | \psi \rangle = F\psi(p_0)$ this yields

$$\rho_{\psi, p_0}(x, p) = \left(\frac{1}{\pi\hbar}\right)^n e^{\frac{2i}{\hbar}(p - p_0)x} \frac{F\psi(2p - p_0)}{F\psi(p_0)}. \quad (11)$$

In view of Eqn. (10) the classical observable $\Pi_x \xrightarrow{\text{Weyl}} \hat{\Pi}_x$ is given by $\Pi_x(x', p') = \delta(x' - x)$ and hence

$$\begin{aligned} \langle \hat{\Pi}_x \rangle_{\text{weak}}^{\phi_{p_0}, \psi} &= \int \rho_{\psi, p_0}(x', p') \delta(x' - x) d^n p' d^n x' \\ &= \int \rho_{\psi, p_0}(x, p') d^n p' \\ &= \frac{\phi_{p_0}^*(x) \psi(x)}{\langle \phi_{p_0} | \psi \rangle} \end{aligned}$$

where the last equality is a consequence of the first marginal distribution property (5); this is precisely the expression (8) of the weak value of $\hat{\Pi}_x$.

We note that the difficulties inherent to the theory of measurement of continuous conjugate variables (such as position and momentum) are real, and far from being resolved. For a mathematically rigorous approach, using the properties of the metaplectic group, see Weigert and Wilkinson [25].

3.2 A general reconstruction formula

Formula (8) shows that we can reconstruct the whole wavefunction ψ by scanning the weak measurements of the projection operator $\hat{\Pi}_x$ through x . We are going to show that, more generally, a quantum state ψ can always be reconstructed from the knowledge of the complex quasi-distribution $\rho_{\phi_p, \psi}$, or, equivalently from the knowledge of the cross-interference term $\mathcal{W}(\phi, \psi)$. Let us begin by introducing some notation. Let $z = (x, p)$ and $z_0 = (x_0, p_0)$ be an arbitrary phase space point, and define the Grossmann–Royer operator [14, 24] $\hat{T}_{\text{GR}}(z_0)$ by

$$\hat{T}_{\text{GR}}(z_0)\psi(x) = e^{\frac{2i}{\hbar}p_0(x-x_0)}\psi(2x_0 - x). \quad (12)$$

It is, up to the complex exponential factor in front of $\psi(2x_0 - x)$ a reflection operator, in fact $\hat{T}_{\text{GR}}(z_0)\hat{T}_{\text{GR}}(z_0) = \hat{I}_d$ (the identity operator). It is in addition a unitary self-adjoint operator: $\hat{T}_{\text{GR}}(z_0)^* = \hat{T}_{\text{GR}}(z_0)^{-1} = \hat{T}_{\text{GR}}(z_0)$. A remarkable fact is that the cross-Wigner transform is related to $\hat{T}_{\text{GR}}(z)$ by the simple formula

$$\mathcal{W}(\phi, \psi)(z) = \left(\frac{1}{\pi\hbar}\right)^n \langle \hat{T}_{\text{GR}}(z)\phi | \psi \rangle \quad (13)$$

(see [10], Chapter 9). Using the following well-known formula (“Moyal identity”, [10], Chapter 9):

$$\int \mathcal{W}(\phi, \psi)^*(z) \mathcal{W}(\phi', \psi')(z) d^{2n}z = \left(\frac{1}{2\pi\hbar}\right)^n \langle \phi | \phi' \rangle \langle \psi | \psi' \rangle \quad (14)$$

and the Grossmann–Royer formalism we prove that the knowledge of the cross-Wigner transform $\mathcal{W}(\phi, \psi)$, and of that one of the two functions ϕ, ψ uniquely determines the other. Moreover, this function can be written in terms of an arbitrary square-integrable auxiliary function γ .

Proposition. Let (ϕ, γ) be a pair of square integrable functions such that $\langle \gamma | \phi \rangle \neq 0$. We have

$$\phi(x) = \frac{2^n}{\langle \psi | \gamma \rangle} \int \mathcal{W}(\phi, \psi)(z_0) \hat{T}_{\text{GR}}(z_0) \gamma(x) d^{2n}z_0 \quad (15)$$

$$\psi(x) = \frac{2^n}{\langle \phi | \gamma \rangle} \int \mathcal{W}(\phi, \psi)^*(z_0) \hat{T}_{\text{GR}}(z_0) \gamma(x) d^{2n}z_0. \quad (16)$$

Proof. We begin with a preliminary remark: both integrals in the formulas above are absolutely convergent. In fact, taking $\phi = \phi'$ and $\psi = \psi'$ in

Moyal's identity (14) we see that $\mathcal{W}(\phi, \psi)$ is square integrable. In view of the Cauchy–Schwarz inequality we have

$$\int |\mathcal{W}(\phi, \psi)(z_0)| |\widehat{T}_{\text{GR}}(z_0)\gamma(x)| d^{2n}z_0 \leq \|\mathcal{W}(\phi, \psi)\|_{L^2} \|\gamma\|_{L^2} < +\infty$$

for the integrals in (15), (16). We next observe that both formulas (15) and (16) are equivalent and obtained from each other by swapping ϕ and ψ and noting that $\mathcal{W}(\psi, \phi)^* = \mathcal{W}(\phi, \psi)$. Let us prove (15). Let us denote by $\chi(x)$ the right hand side of (15):

$$\chi(x) = \frac{2^n}{\langle \psi | \gamma \rangle} \int \mathcal{W}(\phi, \psi)(z_0) \widehat{T}_{\text{GR}}(z_0) \gamma(x) d^{2n}z_0.$$

We are going to show that $\langle \chi | \theta \rangle = \langle \phi | \theta \rangle$ for every element θ of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$; it will follow that we have $\chi = \phi$ almost everywhere, which proves formula (15). We have

$$\langle \chi | \theta \rangle = \frac{2^n}{\langle \psi | \gamma \rangle} \int \mathcal{W}(\phi, \psi)(z) \langle \widehat{T}_{\text{GR}}(z) \gamma | \theta \rangle d^{2n}z.$$

In view of formula (13) we have

$$\langle \widehat{T}_{\text{GR}}(z) \gamma | \theta \rangle = (\pi \hbar)^n \mathcal{W}(\gamma, \theta)(z)$$

and hence

$$\begin{aligned} \langle \chi | \theta \rangle &= \frac{(2\pi \hbar)^n}{\langle \psi | \gamma \rangle} \int \mathcal{W}(\phi, \psi)(z) \mathcal{W}(\gamma, \theta)(z) d^{2n}z \\ &= \frac{(2\pi \hbar)^n}{\langle \psi | \gamma \rangle} \int \mathcal{W}(\psi, \phi)^*(z) \mathcal{W}(\gamma, \theta)(z) d^{2n}z. \end{aligned}$$

Applying Moyal's identity (14) to the last integral we get

$$\int \mathcal{W}(\psi, \phi)^*(z) \mathcal{W}(\gamma, \theta)(z) d^{2n}z = \left(\frac{1}{2\pi \hbar}\right)^n \langle \psi | \gamma \rangle \langle \phi | \theta \rangle$$

and hence $\langle \chi | \theta \rangle = \langle \phi | \theta \rangle$. ■

Since the cross-Wigner transform $\mathcal{W}(\phi, \psi)$ and the weak value $\rho_{\phi, \psi}(x, p)$ are equal up to a factor $\langle \phi | \psi \rangle$ (formula (3)), it follows that we can rewrite (15) and (16) as

$$\phi(x) = 2^n \frac{\langle \phi | \psi \rangle}{\langle \psi | \gamma \rangle} \int \rho_{\phi, \psi}(z_0) \widehat{T}_{\text{GR}}(z_0) \gamma(x) d^{2n}z_0 \quad (17)$$

$$\psi(x) = 2^n \frac{\langle \phi | \psi \rangle^*}{\langle \phi | \gamma \rangle} \int \rho_{\phi, \psi}^*(z_0) \widehat{T}_{\text{GR}}(z_0) \gamma(x) d^{2n}z_0. \quad (18)$$

4 Relation with Time-Frequency Analysis

The Wigner formalism is widely used in time-frequency analysis (TFA), but there is an interpretational and almost philosophical difference in the viewpoints in QM and TFA. While the Wigner formalism is an essential tool for expressing QM in its phase space version and leads to the Weyl quantization scheme [9, 10, 18], the situation is less clear-cut in TFA where the appearance of cross-term correlations as $\mathcal{W}(\phi, \psi)$ is an unwanted artefact; to eliminate or weaken these interference effects one uses elements of the so-called Cohen [7, 13] class (which would typically be the Husimi transform in QM). Everything we have said above can be re-expressed in terms of the cross-ambiguity function familiar from radar theory (Woodward [26], Binz and Pods [6]), and related to the Wigner transform by a symplectic Fourier transform:

$$\mathcal{A}(\phi, \psi)(z) = F_{\sigma} \mathcal{W}(\phi, \psi)(z) = F \mathcal{W}(\phi, \psi)(Jz)$$

where F is here the usual Fourier transform on \mathbb{R}^{2n} and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. It is given by the analytical expression

$$\mathcal{A}(\phi, \psi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py} \phi^*(y + \frac{1}{2}x) \psi(y - \frac{1}{2}x) d^n y.$$

The symplectic Fourier transform being an involution we can rewrite the definition (3) of $\rho_{\phi, \psi}(x, p)$ in the form

$$\rho_{\phi, \psi}(x, p) = \frac{F_{\sigma} \mathcal{A}(\phi, \psi)(x, p)}{\langle \phi | \psi \rangle}. \quad (19)$$

The use of this alternative approach does a priori not bring anything new from a mathematical point of view, but could perhaps be useful for giving interpretations of weak values and measurements in terms of the “source-target” formalism of radar theory.

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